

Final Administrative Report  
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Prof. Dewey H. Hodges, Principal Investigator  
Mr. Matthew Greenman, Supported Student

This grant originally supported Dr. Brian Watson, student of Prof. Manohar P. Kamat. Since some money remained from that original phase of the program, a request was made to transfer the grant to Prof. Dewey H. Hodges who proposed to support Mr. Matthew Greenman with the remaining money and seek renewal to the appropriate time. This was approved by NASA. There were sufficient funds to support Mr. Greenman for two quarters (Summer and Fall 1994). He started to work with the P.I. (the enclosed technical summary reflects his accomplishments), but he became interested in another type of work which, according to him, necessitated his transfer to MIT.

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# Flutter Analysis of Composite Box Beams

Matthew Greenman

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The dynamic aeroelastic instability of flutter is an important factor in the design of modern high-speed, flexible aircraft. The current trend is toward the creative use of composites to delay flutter. To obtain an optimum design, we need a accurate as well as efficient model. As a first step towards this goal, flutter analysis is carried out for an unswept composite box beam using a linear structural model and Theodorsen's unsteady aerodynamic theory. Structurally, the wing was modeled as a thin-walled box-beam of rectangular cross section. Theodorsen's theory was used to get 2-D unsteady aerodynamic forces, which were integrated over the span. A free-vibration analysis is carried out using the theory of Ref. [4]. These fundamental modes are used to get the flutter solution using the V-g method. Future work is intended to build on this foundation.

## Structural Preliminaries

A thin-walled box beam is considered for the analysis. A coordinate system  $\hat{x} = \{\hat{x}_1 \hat{x}_2 \hat{x}_3\}$  is defined along the undeformed wing with  $\hat{x}_1$  along the elastic axis.

The displacements, rotations, strains and curvatures are denoted by  $u$ ,  $\theta$ ,  $\gamma$  and  $\kappa$  respectively

$$u = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \theta = \begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{Bmatrix} \gamma = \begin{Bmatrix} \gamma_{11} \\ 2\gamma_{12} \\ 2\gamma_{13} \end{Bmatrix} \kappa = \begin{Bmatrix} \kappa_1 \\ \kappa_2 \\ \kappa_3 \end{Bmatrix} \quad (1)$$

Transverse shear deformations are negligible, so that  $\gamma_{12}, \gamma_{13} = 0$ . Thus

$$\gamma = \begin{Bmatrix} \gamma_{11} \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} u'_1 \\ 0 \\ 0 \end{Bmatrix} \quad (2)$$

Rotations are small, so that

$$\theta = \begin{Bmatrix} \theta_1 \\ -u'_3 \\ u'_2 \end{Bmatrix} \text{ and } \kappa = \begin{Bmatrix} \theta'_1 \\ -u''_3 \\ u''_2 \end{Bmatrix} \quad (3)$$

Using the theory developed by Badir et.al. (1992) we get a constitutive equation of the form

$$\begin{Bmatrix} F \\ M_1 \\ M_2 \\ M_3 \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} \\ C_{12} & C_{22} & C_{23} & C_{24} \\ C_{13} & C_{23} & C_{33} & C_{34} \\ C_{14} & C_{24} & C_{34} & C_{44} \end{bmatrix} \begin{Bmatrix} u'_1 \\ \theta'_1 \\ -u''_3 \\ u''_2 \end{Bmatrix} \quad (4)$$

where  $F$  is the force along the wing and  $M_1, M_2, M_3$  are the torsional and two bending moments. The  $[C]_{4 \times 4}$  is the cross sectional stiffness matrix and is obtained using the theory.

## Energy Formulation and Raleigh Ritz Method

Langrange equations of motion can be written in terms of the kinetic energy ( $T$ ) and the strain energy ( $U$ ) as

$$\frac{\partial}{\partial t} \left( \frac{\partial(T-U)}{\partial \dot{q}_i} \right) - \frac{\partial(T-U)}{\partial q_i} = Q \quad (5)$$

Kinetic energy is given by

$$T = \frac{1}{2} \int_0^l \left\{ \begin{matrix} \dot{u} \\ \dot{\theta} \end{matrix} \right\}^T \begin{bmatrix} m & -m\tilde{\xi} \\ m\tilde{\xi} & i \end{bmatrix} \left\{ \begin{matrix} \dot{u} \\ \dot{\theta} \end{matrix} \right\} dx_1 \quad (6)$$

where  $m$  is the mass per unit lenth,  $\tilde{\xi}$  is a matrix of center of mass offset from the elastic axis and  $i$  is the matrix of polar moment of inertia. For our problem Eq. (6) becomes

$$T = \frac{1}{2} \int_0^l \left\{ \begin{matrix} \dot{u}_1 \\ \dot{\theta}_1 \\ -\dot{u}_3 \\ \dot{u}_2 \end{matrix} \right\}^T \begin{bmatrix} m & 0 & 0 & 0 \\ 0 & i & -m\tilde{\xi}_2 & -m\tilde{\xi}_3 \\ 0 & -m\tilde{\xi}_2 & m & 0 \\ 0 & -m\tilde{\xi}_3 & 0 & m \end{bmatrix} \left\{ \begin{matrix} \dot{u}_1 \\ \dot{\theta}_1 \\ -\dot{u}_3 \\ \dot{u}_2 \end{matrix} \right\} dx_1 \quad (7)$$

Strain energy is given by

$$U = \frac{1}{2} \int_0^l \left\{ \begin{matrix} \gamma \\ \kappa \end{matrix} \right\}^T [C] \left\{ \begin{matrix} \gamma \\ \kappa \end{matrix} \right\} dx_1 \quad (8)$$

so that

$$U = \frac{1}{2} \int_0^l \left\{ \begin{matrix} u'_1 \\ \theta'_1 \\ -u''_3 \\ u''_2 \end{matrix} \right\}^T \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} \\ C_{12} & C_{22} & C_{23} & C_{24} \\ C_{13} & C_{23} & C_{33} & C_{34} \\ C_{14} & C_{24} & C_{34} & C_{44} \end{bmatrix} \left\{ \begin{matrix} u'_1 \\ \theta'_1 \\ -u''_3 \\ u''_2 \end{matrix} \right\} dx_1 \quad (9)$$

Now generalized coordinates are introduced to make it a finite dimensional system. We represent the deflections  $u$  and  $\theta$  with finite number of modeshapes yielding

$$\left\{ \begin{matrix} u_1 \\ \theta_1 \\ -u_3 \\ u_2 \end{matrix} \right\} = [\Phi] \{q\} \quad (10)$$

where

$$[\Phi] = \begin{bmatrix} \{\phi_{u_1}\}_{1 \times n} & 0 & 0 & 0 \\ 0 & \{\phi_{\theta_1}\}_{1 \times o} & 0 & 0 \\ 0 & 0 & \{\phi_{u_3}\}_{1 \times p} & 0 \\ 0 & 0 & 0 & \{\phi_{u_2}\}_{1 \times q} \end{bmatrix} \quad (11)$$

Here  $n, o, p, q$  are the number of extentional, torsional, vertical bending, and inplane bending modes, respectively. Thus, we also have

$$\left\{ \begin{matrix} u'_1 \\ \theta'_1 \\ -u''_3 \\ u''_2 \end{matrix} \right\} = [\Phi^*] \{q\} \quad (12)$$

where

$$[\Phi^*] = \begin{bmatrix} \{\phi'_{u_1}\}_{1 \times n} & 0 & 0 & 0 \\ 0 & \{\phi'_{\theta_1}\}_{1 \times o} & 0 & 0 \\ 0 & 0 & \{\phi''_{u_3}\}_{1 \times p} & 0 \\ 0 & 0 & 0 & \{\phi''_{u_2}\}_{1 \times q} \end{bmatrix} \quad (13)$$

Substituting Eq. (10) and Eq. (12) in the kinetic and strain energy expressions given by Eq. (7) and Eq. (9), the Lagrange's equations can be expressed as

$$[M]\{\ddot{q}\} + [K]\{q\} = \{Q\} \quad (14)$$

where

$$[M] = \int_0^l [\Phi]^T \begin{bmatrix} m & 0 & 0 & 0 \\ 0 & i & -m\tilde{\xi}_2 & -m\tilde{\xi}_3 \\ 0 & -m\tilde{\xi}_2 & m & 0 \\ 0 & -m\tilde{\xi}_3 & 0 & m \end{bmatrix} [\Phi] dx_1 \quad (15)$$

$$[K] = \int_0^l [\Phi^*]^T \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} \\ C_{12} & C_{22} & C_{23} & C_{24} \\ C_{13} & C_{23} & C_{33} & C_{34} \\ C_{14} & C_{24} & C_{34} & C_{44} \end{bmatrix} [\Phi^*] dx_1 \quad (16)$$

Calculation of  $\{Q\}$  will be discussed later.

For the present analysis,  $n$  extensional,  $o$  torsional,  $p$  vertical bending and  $q$  inplane bending uncoupled modes are taken as  $\{\phi_{u_1}\}$ 's,  $\{\phi_{\theta_1}\}$ 's,  $\{\phi_{u_3}\}$ 's and  $\{\phi_{u_2}\}$ 's respectively. Analytic expressions for the modes were taken from Ref. [2]. Analytic expressions for the above integrals were derived for the case of uniform cross section beams.

## Free vibration analysis

For free vibration analysis,  $\{Q\} \equiv 0$ , thus Eq. (14) becomes

$$[M]\{\ddot{q}\} + [K]\{q\} = 0 \quad (17)$$

Now assuming simple harmonic motion, i.e.  $\{q\} = \{\bar{q}\}e^{i\omega t}$ , we get an eigenvalue problem, whose eigenvalues give the frequencies and eigenvectors the modeshapes of vibration.

$$[K]\{\bar{q}\} = \omega^2[M]\{\bar{q}\} \quad (18)$$

Given below is a comparison of free-vibration frequencies of a Bending-Twist Coupled Beam with experimental values. A graphite epoxy cantilever plate beam is considered. The results validate the free-vibration analysis code.

Mode	Experimental	Numerical
1B	3.6	3.6
2B	24.2	22.6
3B	66.5	63.3
4B	124	124

## Coordinate Transformations

Aeroelastic analyses using simple structural models have traditionally relied upon aerodynamic variables such as angle of attack ( $\alpha$ ) and plunging motion ( $h$ ) to describe the aerodynamic forces on the airfoil. Because the present analysis incorporates a more sophisticated structural model, it was found to be more convenient to express the aerodynamic parameters in terms of the wing kinematic variables.

The wing section has an angle of attack  $\alpha$  in a uniform freestream of velocity  $U$ . The reference frame  $\hat{a}$  is oriented such that  $\vec{U} = -U\hat{a}_2$  (see Figure 1). The reference frame  $\hat{b}$  is oriented with respect to the undeformed wing section and is centered at the section reference line. Thus, it is the same as the structural coordinate system  $\hat{x}$ , i.e.  $\hat{b}_i \equiv \hat{x}_i$ . The reference frame  $\hat{B}$  is a small angle transformation with respect to  $\hat{b}$  caused by deformation of the section such that

$$\begin{Bmatrix} \hat{B}_1 \\ \hat{B}_2 \\ \hat{B}_3 \end{Bmatrix} = \begin{bmatrix} 1 & u'_2 & u'_3 \\ -u'_2 & 1 & \theta_1 \\ -u'_3 & -\theta_1 & 1 \end{bmatrix} \begin{Bmatrix} \hat{b}_1 \\ \hat{b}_2 \\ \hat{b}_3 \end{Bmatrix} \quad (19)$$

The section of the undeformed beam is rotated at a steady-state angle of attack  $\alpha_o$  from the freestream such that the transformation from  $\hat{a}$  to  $\hat{b}$  is

$$\begin{Bmatrix} \hat{b}_1 \\ \hat{b}_2 \\ \hat{b}_3 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha_o & \sin \alpha_o \\ 0 & -\sin \alpha_o & \cos \alpha_o \end{bmatrix} \begin{Bmatrix} \hat{a}_1 \\ \hat{a}_2 \\ \hat{a}_3 \end{Bmatrix} \quad (20)$$

In this analysis, we assume that the steady-state angle of attack is small (i.e. the same order of magnitude as the angles in Eq. 19). This allows us to make the approximations

$$\cos \alpha_o = 1 \quad \text{and} \quad \sin \alpha_o = \alpha_o. \quad (21)$$

Combining Eq. (19), Eq. (20) and Eq. (21), and discarding the higher order terms, the transformation between the freestream reference frame and the deformed beam sectional reference frame is

$$\begin{Bmatrix} \hat{B}_1 \\ \hat{B}_2 \\ \hat{B}_3 \end{Bmatrix} = \begin{bmatrix} 1 & u'_2 & u'_3 \\ -u'_2 & 1 & \alpha_o + \theta_1 \\ -u'_3 & -(\alpha_o + \theta_1) & 1 \end{bmatrix} \begin{Bmatrix} \hat{a}_1 \\ \hat{a}_2 \\ \hat{a}_3 \end{Bmatrix} \quad (22)$$

This transformation shows the section angle of attack as a function of time to be

$$\alpha(t) = \alpha_o + \theta_1(t) \quad (23)$$

The deflections  $u_i$  along the undeformed reference axes  $\hat{b}_i$  are also expressed in terms of the freestream reference frame  $\hat{a}$ . Carrying through the small angles assumption of Eq. (21) and assuming the displacements also to be small, it can be verified that

$$u_i \hat{x}_i = u_i \hat{b}_i \approx u_i \hat{a}_i \quad (24)$$

This is a gross approximation, but it is allowed in this analysis because the small angle and small displacement assumptions are within the limitations of Theodorsen's theory. From Eq. (24), the section plunging motion is

$$h = -u_3. \quad (25)$$

## Development of Generalized Aerodynamic Forces

Lagrange's equations were developed from kinetic and strain energy formulations for a uniform, cantilevered, thin-walled, closed-section beam and are given by

$$M\ddot{q} + Kq = Q \quad (26)$$

where  $M$  is the  $n \times n$  generalized mass matrix,  $K$  is the  $n \times n$  generalized stiffness matrix,  $Q$  is the  $n \times 1$  generalized force vector, and  $q$  are the generalized coordinates. There are two components of the generalized forces that must be considered in the flutter analysis: the aerodynamic loadings and the structural damping. This analysis uses 2-dimensional incompressible strip theory to obtain the aerodynamic loadings. The effect of structural damping is introduced artificially using the V-g method, as will be discussed in a later section.

The vector of generalized aerodynamic forces is developed in from the principle of virtual work. The virtual work done on a two-dimensional airfoil section is expressed as:

$$\delta\overline{W} = (\vec{L} + \vec{D}) \cdot \delta\vec{r} + \vec{M} \cdot \delta\vec{\Psi} \quad (27)$$

where  $\vec{L}$ ,  $\vec{D}$  and  $\vec{M}$  are the section lift, drag and moment, respectively;  $\delta\vec{r}$  is the virtual displacement vector, and  $\delta\vec{\Psi}$  is the virtual rotation vector:

$$\delta\vec{r} = \delta u_i \hat{a}_i + \delta\vec{\Psi} \times (ba\hat{B}_2) \quad \delta\vec{\Psi} = \{ \delta\theta_1 \quad -\delta u'_3 \quad \delta u'_2 \} \begin{Bmatrix} \hat{B}_1 \\ \hat{B}_2 \\ \hat{B}_3 \end{Bmatrix} \quad (28)$$

The aerodynamic loadings are calculated with respect to the freestream reference frame  $\hat{a}$  and about the midchord, such that  $\vec{L} = L\hat{a}_3$ ,  $\vec{D} = -D\hat{a}_2$  and  $\vec{M} = M\hat{a}_1$ . The reference line offset from the midchord is  $ba\hat{B}_2$ . The virtual work can be rewritten using Eq. (28) in terms of the virtual displacements and the virtual rotation about the reference line:

$$\delta\overline{W} = P^T \delta u + R^T \delta u' \quad (29)$$

where  $P$  and  $R$  are  $4 \times 1$  column matrices and

$$\delta u = \begin{Bmatrix} \delta u_1 \\ \delta\theta_1 \\ -\delta u_3 \\ \delta u_2 \end{Bmatrix} \quad \text{and} \quad \delta u' = \begin{Bmatrix} \delta u'_1 \\ \delta\theta'_1 \\ -\delta u'_3 \\ \delta u'_2 \end{Bmatrix}$$

The virtual work can be expressed in terms of generalized coordinates, using the expansion

$$u = \Phi q \quad (30)$$

where  $\Phi$  is the modal matrix and  $q$  is the generalized coordinate vector. The virtual work in terms of generalized coordinates is

$$\delta\overline{W} = P^T \frac{\partial u}{\partial q} \delta q + R^T \frac{\partial u'}{\partial q} \delta q$$

or, from Eq. (30),

$$\delta\overline{W} = (P^T \Phi + R^T \Phi') \delta q$$

The coefficient of  $\delta q$  is the transpose of the generalized aerodynamic forces on the section. Integrating over the wing span, we obtain the generalized aerodynamic forces as

$$Q = \int_0^L (\Phi^T P + \Phi'^T R) dx \quad (31)$$

The elements of  $P$  and  $R$  are, in terms of the lift, drag and moment,

$$\begin{aligned} P_1 &= 0 & R_1 &= 0 \\ P_2 &= M + baL + baD(\alpha_o + \theta_1) & R_2 &= 0 \\ P_3 &= -L & R_3 &= -Mu'_2 \\ P_4 &= -D & R_4 &= baDu'_2 - baLu'_3 - Mu'_3 \end{aligned} \quad (32)$$

## 2-Dimensional Strip Theory (Theodorsen)

Expressions for the unsteady aerodynamic lift and moment on an airfoil were initially developed by Theodorsen. Assuming simple harmonic motion the total lift and moment can be expressed as

$$\begin{aligned} L &= L_0 + \bar{L}e^{i\omega t} \\ M &= M_0 + \bar{M}e^{i\omega t} \end{aligned} \quad (33)$$

here  $L_0$  and  $M_0$  are the steady components of the lift and moment associated with the steady-state angle of attack  $\alpha_o$ ;  $\bar{L}$  and  $\bar{M}$  are the complex magnitudes of the unsteady lift and moment, respectively; and  $\omega$  is the natural frequency. The steady components of the aerodynamic loadings are given from Ref. [1] in terms of reduced frequency  $k = \omega b/U$  as

$$\begin{aligned} L_0 &= \frac{\rho b^3 \omega^2}{k^2} c_{l_o} \\ M_0 &= \frac{\rho b^4 \omega^2}{k^2} c_{m_o} \end{aligned} \quad (34)$$

In these equations,  $\rho$  is the air density and  $b$  is the semichord. Because this flutter analysis is but a first attempt using a primitive aerodynamic theory, the steady lift and moment coefficients are assumed to be those of a thin symmetric airfoil, i.e.

$$c_{l_o} = 2\pi\alpha_o \quad \text{and} \quad c_{m_o} = \pi\alpha_o \quad (35)$$

The unsteady components of the lift and moment are derived from the complex magnitudes of the lift and moment coefficients in Ref. [1] for the case of simple harmonic pitching and plunging motions. These unsteady loadings are:

$$\begin{aligned} \bar{L} &= -\pi\rho b^3\omega^2 \left\{ L_h \frac{\bar{h}}{b} + [L_\alpha - (\frac{1}{2} + a)L_h]\bar{\alpha} \right\} \\ \bar{M} &= \pi\rho b^4\omega^2 \left\{ [M_h - \frac{1}{2}L_h]\frac{\bar{h}}{b} + [M_\alpha - \frac{1}{2}L_\alpha - (\frac{1}{2} + a)(M_h - \frac{1}{2}L_h)]\bar{\alpha} \right\} \end{aligned} \quad (36)$$

The coefficients  $L_\alpha$ ,  $L_h$ ,  $M_\alpha$  and  $M_h$  are functions of reduced frequency  $k$  and the Theodorsen function  $C(k)$ , and are given from Ref. [1] as

$$\begin{aligned} L_\alpha &= \frac{1}{2} - \frac{i}{k}(1 + 2C(k)) - \frac{2}{k^2}C(k) & M_\alpha &= \frac{3}{8} - \frac{i}{k} \\ L_h &= 1 - \frac{2i}{k}C(k) & M_h &= \frac{1}{2} \end{aligned} \quad (37)$$

The total lift and moment can be written in terms of the kinematic variables by recognizing that, from Eq. (23) and Eq. (25)

$$\bar{h} = -\bar{u}_3 \quad \bar{\alpha} = \bar{\theta}_1$$

Also, in accord with the assumption of simple harmonic motion,

$$u_3 = \bar{u}_3 e^{i\omega t} \quad \theta_1 = \bar{\theta}_1 e^{i\omega t}$$

Expressions for the total lift and moment are thus

$$\begin{aligned} L &= \frac{2\pi\rho b^3\omega^2}{k^2}\alpha_o - \pi\rho b^3\omega^2\left\{-L_h\frac{u_3}{b} + [L_\alpha - (\frac{1}{2} + a)L_h]\theta_1\right\} \\ M &= \frac{\pi\rho b^4\omega^2}{k^2}\alpha_o + \pi\rho b^4\omega^2\left\{[M_h - \frac{1}{2}L_h]\frac{-u_3}{b} \right. \\ &\quad \left. + [M_\alpha - \frac{1}{2}L_\alpha - (\frac{1}{2} + a)(M_h - \frac{1}{2}L_h)]\theta_1\right\} \end{aligned} \quad (38)$$

The drag on the airfoil section is developed by introducing the small perturbations in velocity and angle of attack into the equation

$$D = \rho b U^2 c_D(\alpha) \quad (39)$$

From the geometry and noting the approximations made in Eq. (24) and Eq. (23), the velocity is  $U + \dot{u}_2$  and the angle of attack is  $\alpha = \alpha_o + \theta_1$ . The velocity  $U$  is expressed in terms of reduced frequency  $k$ . In addition, it is recognized that, for simple harmonic motion,

$$\dot{u}_2 = i\omega\bar{u}_2 e^{i\omega t} = i\omega u_2$$

Substituting these perturbations into Eq. (39) and using a Taylor series expansion, the linear approximation of the unsteady section drag is:

$$D = \frac{\rho b^3\omega^2}{k^2}c_D(\alpha_o) + \frac{\rho b^3\omega^2}{k^2}\frac{dc_D}{d\alpha}\bigg|_{\alpha_o}\theta_1 + \frac{2i\rho b^2\omega^2}{k}c_D(\alpha_o)u_2 \quad (40)$$

The generalized unsteady aerodynamic forces are obtained by substituting Eq. (38) and Eq. (40) into the relations for  $P$  and  $R$  in Eq. (32), and subsequently integrating Eq. (31). Neglecting higher-order terms, the non-zero elements of  $P$  and  $R$  become

$$\begin{aligned} P_2 &= \pi\rho b^4\omega^2\left[-\left(\frac{1}{2} + a\right)L_\alpha + \left(\frac{1}{4} + a + a^2\right)L_h \right. \\ &\quad \left. + M_\alpha - \left(\frac{1}{2} + a\right)M_h + \frac{a}{\pi k^2}\left(c_D(\alpha_o) + \frac{dc_D}{d\alpha}\bigg|_{\alpha_o}\alpha_o\right)\right]\theta_1 \\ &\quad + \pi\rho b^3\omega^2\left[\left(\frac{1}{2} + a\right)L_h - M_h\right]u_3 + \frac{2i\rho ab^3\omega^2 c_D(\alpha_o)\alpha_o}{k}u_2 \\ &\quad + \frac{\pi\rho b^4\omega^2}{k^2}\left[1 + 2a + \frac{ac_D(\alpha_o)}{\pi}\right]\alpha_o \\ P_3 &= \pi\rho b^3\omega^2\left[L_\alpha - \left(\frac{1}{2} + a\right)L_h\right]\theta_1 - \pi\rho b^2\omega^2 L_h u_3 - \frac{2\pi\rho b^3\omega^2}{k^2}\alpha_o \end{aligned} \quad (41)$$

$$\begin{aligned}
P_4 &= -\frac{\rho b^3 \omega^2}{k^2} \frac{dc_D}{d\alpha} \Big|_{\alpha_o} \theta_1 - \frac{2i\rho b^2 \omega^2 c_D(\alpha_o)}{k} u_2 - \frac{\rho b^3 \omega^2 c_D(\alpha_o)}{k^2} \\
R_3 &= -\frac{\pi \rho b^4 \omega^2 \alpha_o}{k^2} u'_2 \\
R_4 &= -\frac{\pi \rho b^4 \omega^2 \alpha_o}{k^2} (1 + 2a) u'_3 + \frac{\rho a b^4 \omega^2 c_D(\alpha_o)}{k^2} u'_2
\end{aligned}$$

The natural frequency term  $\omega^2$  can be factored out of each term in the above expressions, and these forces can be rewritten in the matrix form

$$P = \omega^2 A u + \omega^2 B \quad R = \omega^2 D u' \quad (42)$$

The  $4 \times 4$  matrices  $A$  and  $D$  and the  $4 \times 1$  column vector  $B$  are shown in the appendix. Using the modal expansion in equation (30) and substituting the above expressions into Eq. (31), the generalized aerodynamic forces become:

$$Q = \omega^2 \left\{ \int_0^L (\Phi^T A \Phi + \Phi'^T D \Phi') dx \right\} q + \omega^2 \int_0^L \Phi^T B dx \quad (43)$$

## Equations of Motion

Analysis of the flutter problem is greatly simplified when the mass matrix  $M$  in Eq. (26) is a diagonal matrix. The mass matrix in our analysis, however, contains off-diagonal terms that account for inertial couplings in the beam that arise when the center of mass does not coincide with the reference line. To solve this problem, the equations of motion in Eq. (26) can be recast in terms of a diagonalized mass matrix. To do this we consider the problem of the wing in free vibration

$$M \ddot{q} + K q = 0 \quad (44)$$

The eigenvectors  $\bar{q}_i$  and eigenvalues  $\omega_i^2$  of this problem are obtained and used to simplify the equations of motion in the flutter problem. The mass and stiffness matrices can be pre- and post-multiplied by the  $n \times n$  eigenvector matrix  $\Upsilon$ , where

$$\Upsilon = [\bar{q}_1 \quad \bar{q}_2 \quad \dots \quad \bar{q}_n]$$

such that

$$\Upsilon^T M \Upsilon \ddot{q} + \Upsilon^T K \Upsilon q = 0$$

The matrix  $M^D = \Upsilon^T M \Upsilon$  is the diagonalized mass matrix, and the matrix  $K^D = \Upsilon^T K \Upsilon$  is the diagonalized stiffness matrix. From the solution of equation (44), it can be seen that

$$K^D = \omega_R^2 \Omega M^D$$

Here we have introduced a reference frequency  $\omega_R$  that will ultimately simplify the flutter equation. This reference frequency is arbitrary, and is traditionally taken to be the natural frequency of the wing in the first torsion mode. The  $n \times n$  matrix  $\Omega$  contains the free vibration natural frequencies in the following form:

$$\Omega = \begin{bmatrix} \left(\frac{\omega_1}{\omega_R}\right)^2 & 0 & \dots & 0 \\ 0 & \left(\frac{\omega_2}{\omega_R}\right)^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \left(\frac{\omega_n}{\omega_R}\right)^2 \end{bmatrix}$$

This diagonalization technique is applied to the equations of motion for the flutter problem. Using the eigenvectors and eigenvalues from the free vibration problem of Eq. (44), Lagrange's equations (26) become

$$M^D \ddot{q} + \omega_R^2 \Omega M^D q - \omega^2 \Xi q = \omega^2 \Upsilon^T \int_0^L \Phi^T B dx \quad (45)$$

where  $\Xi$  is the generalized aerodynamic forces from Eq. (43) premultiplied by the transpose of the eigenvector matrix such that

$$\Xi = \Upsilon^T \int_0^L (\Phi^T A \Phi + \Phi^T D \Phi) dx$$

## Structural Damping

As pointed out in Bisplinghoff, Ashley and Halfman [1], a flutter analysis cannot provide accurate results without accounting for the effects of structural damping in some fashion. In flutter specifically, structural damping is the only mechanism through which energy is removed from the wing. A precise description of the structural damping in the wing is difficult to obtain; however, since it is small compared to the aerodynamic loadings, it is possible to approximate the structural damping by an artificial damping force. To characterize this artificial damping force, consider a single degree of freedom system undergoing the simple harmonic motion

$$x = x_0 e^{i\omega t}$$

A viscous damping force opposing this motion could be

$$F_D = -f \frac{dx}{dt} = -f i \omega x_0 e^{i\omega t}$$

The viscous damping force in this example lags the position vector by 90° and has an amplitude that depends on the frequency. The structural damping in the flutter problem is known to be frequency independent, so the viscous damping model is not a valid approximation in toto. It does show, however, the presence of a phase shift between the motion and the corresponding damping force.

In the motion of a beam element in free vibration, an artificial structural damping can be associated with each elastic restoring force and torque by multiplying the restoring force by a small, artificial damping coefficient  $g_i$  associated with each motion. In terms of generalized coordinates, this artificial damping force can be defined from Eq. (44) as

$$\Xi_i^s = -g_i \frac{M_i^D}{\omega} \omega_i^2 \dot{q}_i$$

or, in matrix form,

$$\Xi^s = -iG \frac{M_i^D}{\omega} \omega_i^2 \dot{q} \quad (46)$$

where

$$G = \begin{bmatrix} g_1 & 0 & \dots & 0 \\ 0 & g_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & g_n \end{bmatrix}$$

The coefficients  $g_i$  have absolutely no physical meaning. The flutter condition occurs when one of these coefficients goes to zero in the flutter equation. Because the structural damping effects are small in comparison with the aerodynamic forces, we can for the purpose of this analysis assume that

$$g_1 \simeq g_2 \simeq \dots \simeq g_n \simeq g$$

The individual values of  $g$  will be recovered from the solution of the flutter determinant, as will be seen.

## Flutter Equation

The generalized artificial damping force vector can now be included in Eq. (45) to obtain the complete equations of motion for a wing in flutter.

$$M^D \ddot{q} + \frac{ig}{\omega} \omega_R^2 \Omega M^D \dot{q} + \omega_R^2 \Omega M^D q - \omega^2 \Xi q = \omega^2 \Upsilon^T \int_0^L \Phi^T B dx \quad (47)$$

This is an inhomogeneous ordinary differential equation in  $q$ . The inhomogeneous term can be dropped if we consider the generalized displacement to be composed of steady-state and time-dependent components, such that, assuming again simple harmonic motion,

$$q = q_o + \bar{q} e^{i\omega t}$$

Dividing the remaining terms by the natural frequency  $\omega^2$  and rearranging, the homogeneous flutter equation becomes

$$\left[ \Omega M^D \left( \frac{\omega_R^2}{\omega^2} \right) (1 + ig) - M^D - \Xi \right] \bar{q} = 0 \quad (48)$$

The flutter conditions are found by setting the flutter determinant to zero for a range of reduced frequencies  $k_R$  and plotting the natural frequency  $\omega$  and artificial damping coefficient  $g$  with respect to the reduced velocity,  $1/k_R$ . Flutter occurs at the reduced velocity where  $g$  goes to zero. The flutter determinant is

$$\det \left[ \Omega M^D \left( \frac{\omega_R^2}{\omega^2} \right) (1 + ig) - M^D - \Xi \right] \quad (49)$$

The components of the flutter determinant can be arranged in a convenient form that allows the determinant to be solved in terms of a single complex polynomial. This is done by introducing the parameter  $\omega_R^2$ , which is a given natural frequency of vibration for a specific mode (usually taken to be the first torsion mode). The components of this determinant are:

$$M_i^D \left[ \left( \frac{\omega_i}{\omega_R} \right)^2 Z - \left( 1 + \frac{\Xi_{ii}}{M_i^D} \right) \right] \quad \text{for } i = j \\ -\Xi_{ij} \quad \text{for } i \neq j \quad (50)$$

where

$$Z = \left( \frac{\omega_R}{\omega} \right)^2 (1 + ig)$$

The natural frequency  $\omega$  and the artificial damping  $g$  are recovered from  $Z$  for each given reduced frequency.

## Preliminary Results and Future Research

A flutter analysis was carried out for a Uniform Cantilever Beam. The result is compared with that obtained by Goland Ref. [3]. The flutter velocity obtained by the code written was 305 mph while Ref. [3] obtained 385mph. The discrepancy in the result may be attributed to some differences in the aerodynamic model used and some slight numerical errors in Ref. [3]. The formulation developed is quite general and can form basis for further work in this field. The structural model could be extended to include nonlinearities. The aerodynamic model could be updated to a more accurate doublet lattice method. To do a time-domain analysis for accurate results away from the flutter point, a finite-state aerodynamics could be utilized.

## Appendix

The elements of the matrices  $A$ ,  $B$  and  $D$  contained in equation (42) are presented below.

Matrix  $A$ :

$$\begin{aligned}
 A_{11} &= A_{12} = A_{13} = A_{14} = 0 \\
 A_{21} &= A_{31} = A_{41} = 0 \\
 A_{34} &= A_{43} = 0 \\
 A_{22} &= \pi \rho b^4 \left[ -\left(a + \frac{1}{2}\right) L_\alpha + \left(\frac{1}{4} + a + a^2\right) L_h + M_\alpha \right. \\
 &\quad \left. - \left(\frac{1}{2} + a\right) M_h + \frac{a}{\pi k^2} \left( c_D(\alpha_o) + \frac{dc_D}{d\alpha} \Big|_{\alpha_o} \alpha_o \right) \right] \\
 A_{23} &= -\pi \rho b^3 [(1 + 2a) L_h - M_h] \\
 A_{24} &= \frac{2i \rho a b^3 c_D(\alpha_o) \alpha_o}{k} \\
 A_{32} &= \pi \rho b^3 \left[ L_\alpha - \left(\frac{1}{2} + a\right) L_h \right] \\
 A_{33} &= \pi \rho b^2 L_h \\
 A_{42} &= -\frac{\rho b^3}{k^2} \frac{dc_D}{d\alpha} \Big|_{\alpha_o} \\
 A_{44} &= -\frac{2i \rho b^2 c_D(\alpha_o)}{k}
 \end{aligned}$$

Matrix  $B$ :

$$\begin{aligned}
 B_1 &= 0 \\
 B_2 &= \frac{\pi \rho b^4}{k^2} \left[ 1 + 2a + \frac{a c_D(\alpha_o)}{\pi} \right] \alpha_o \\
 B_3 &= -\frac{2\pi \rho b^3}{k^2} \alpha_o \\
 B_4 &= -\frac{\rho b^3 c_D(\alpha_o)}{k^2}
 \end{aligned}$$

Matrix  $D$ :

$$\begin{aligned}
 D_{11} &= D_{12} = D_{13} = D_{14} = 0 \\
 D_{21} &= D_{22} = D_{23} = D_{24} = 0
 \end{aligned}$$

$$\begin{aligned}
D_{31} &= D_{32} = D_{33} = D_{34} = 0 \\
D_{41} &= D_{42} = 0 \\
D_{34} &= -\frac{\pi \rho b^4 \alpha_o}{k^2} \\
D_{43} &= -\frac{\pi \rho b^4 \alpha_o}{k^2} (1 + 2a) \\
D_{44} &= -\frac{\rho a b^4 c_D(\alpha_o)}{k^2}
\end{aligned}$$

## References

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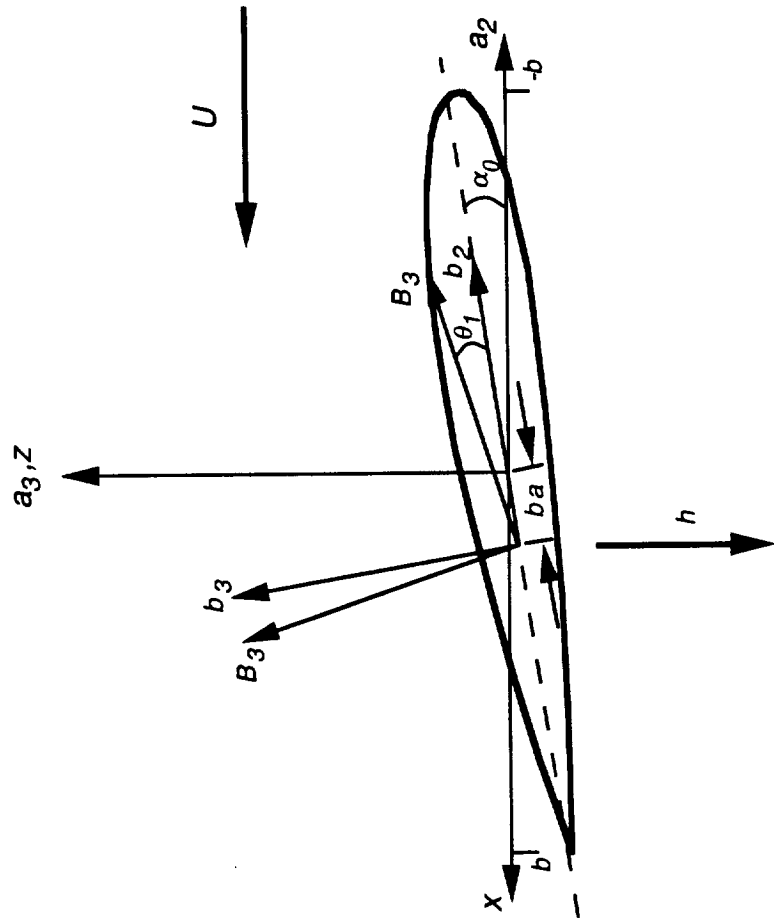


Figure 1